Structure of the Quark Propagator at High Temperature

H. Arthur Weldon
Department of Physics, West Virginia University, Morgantown, West Virginia, 26506
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In the high temperature, chirally invariant phase of QCD, the quark propagator is shown to have two sets of poles with different dispersion relations. A reflection property in momentum space relates all derivatives at zero-momentum of the particle and hole energies, the particle and hole damping rates, and the particle and hole residues. No use is made of perturbation theory.

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1. Background

In the high temperature, chirally invariant phase of gauge theories the fermion propagator has some unusual properties in the one-loop approximation. Despite explicit chiral invariance the fermion has an effective mass proportional to temperature [1,2]. The "mass" is a consequence of the preferred reference frame of the heat bath, which allows the pole of the fermion propagator to be off the light cone. The second and more surprising effect is that there are two poles in the propagator corresponding to two different dispersion relations, both with positive energy [1,3,4]. At zero momentum the two branches coincide. The branch describing normal particle excitations increases monotonically with momentum. The other branch is a collective excitation, referred to either as a hole or as a plasmino, that decreases slightly at small momentum, reaches an absolute minimum, and then rises. The excitations on the upper branch of the dispersion curve have the same chirality and helicity (i.e. both positive or both negative) as is customary for particles. The hole excitations of the lower branch have chirality opposite to helicity. These results are reviewed by Le Bellac [5].

In the hard thermal loop approximation of Braaten and Pisarski, the effective fermion propagator is the starting point for consistent higher order calculations [6–11]. The doubled fermion dispersion relations have been important in calculations of dilepton production [12–14] and strangeness production [15] by a quark-gluon plasma. There have been investigations of how the dispersion relations are affected by retaining non-leading powers of temperature [16,17], by including a bare mass [18,19], and by a including a chemical potential [20–22].

In the electroweak sector at high temperature the quarks, charged leptons, and neutrinos all have effective thermal masses and doubled dispersion relations [2,23–27]. The phenomena also occurs with Majorana fermions [28]. Farrar and Shaposhnikov exploited the importance of chirality transport in baryogenesis at the electroweak scale and used the doubled fermion dispersion relations to account for the baryon asymmetry of the universe [29]. However, it has been shown that including fermion damping makes the effect too small [30–33]. There are renewed attempts to account for baryogenesis at the electroweak scale using the fermion dispersion relations in models with two Higgs doublets [34] and in minimal supersymmetric models [35].

2. QCD Motivation

The evidence discussed thus far for the thermal fermion mass and for the doubled dispersion relations is based entirely on one-loop perturbation theory. The one-loop calculations are sufficiently accurate for electroweak effects but not for QCD phenomenology. Existing calculations are only valid if g is very small. Unfortunately, the most interesting aspects of the quark dispersion relations are limited both by the small coupling and also by some accidentally small coefficients. For example, the thermal mass is $m = 0.41gT$. The minimum in the dispersion relation for the hole excitation is slightly below this at $E = 0.38gT$ and the minimum occurs at a very small momentum, $p = 0.17gT$. At temperatures well above the critical temperature g is indeed small, but near T_c it is not.

There is some nonperturbative evidence that the one-loop calculations are qualitatively correct. Peshier et al [36,37] and also Lévai and Heinz [38] have successfully fit lattice simulations of high temperature QCD using effective quark masses [36–38]. These studies do not test the doubled dispersion relation. Recently Schäfer and Thoma have computed the quark self-energy in the presence of a gluon condensate [39]. Their calculation is one-loop but with the gluon condensate determining the gluon propagator. In their calculation it is the soft loop momenta that control the quark dispersion relation rather than the hard momenta as is usually the case. Nevertheless the results are qualitatively similar in several respects. There is an effective mass, but now related to the condensates $\langle\vec{E}^2\rangle$ and $\langle\vec{B}^2\rangle$. Using lattice data for the condensates gives $m \approx 1.15T$ for temperatures in the range $1.1T_c \leq T \leq 4T_c$. There are, in addition, two different dispersion relations. They coincide at $p = 0$ but at small p have opposite slopes. The dispersion relation for the quark is monotonically increasing; that for the hole decreases slightly, reaching a minimum at $p \approx m/2$ and

then increasing at larger p. The self-energy computed by Schäfer and Thoma is a polynomial in p_0 and p divided by $(p_0^2 - p^2)^3$, whereas the hard thermal loop self-energy contains $\ln(p_0 \pm p)$.

This naturally leads to the question of whether the existence of the separate particle and hole dispersion relation are independent of weak coupling and of the one-loop approximation. This paper will demonstrate that, independently of perturbation theory, the fermion propagator will always have two distinct dispersion relations in the high temperature, chirally symmetric phase.

3. Retarded propagator

It will be most convenient to analyze the retarded propagator and later express the time-ordered propagator in terms of it. The free retarded propagator is

$$S^R_{\rm free}(p_0,\vec{p}) = \frac{\gamma_0 p_0 - \vec{\gamma}\cdot\vec{p}}{(p_0+i\eta)^2 - \vec{p}^2},$$

and is independent of temperature. It can be rewritten as

$$S^R_{\rm free}(p_0,\vec{p}) = \frac{1}{2}\frac{\gamma_0 - \vec{\gamma}\cdot\hat{p}}{p_0 - p + i\eta} + \frac{1}{2}\frac{\gamma_0 + \vec{\gamma}\cdot\hat{p}}{p_0 + p + i\eta}. \qquad (1)$$

The particle pole is in the fourth quadrant of the complex p_0 plane at $p_0 = p - i\eta$; the antiparticle pole is in the third quadrant at $p_0 = -p - i\eta$.

The full retarded propagator has both poles and cuts but the singularities are always in the lower half-plane. At nonzero temperature the propagator in the rest frame of the heat bath will depend separately on energy p_0 and momentum $\vec{p}$. Invariance under chirality and parity limits the self-energy to be a linear combination of the matrices γ_0 and $\vec{\gamma}\cdot\vec{p}$. The most general possibility can be written

$$S^R(p_0,\vec{p}) = \frac{\frac{1}{2}(\gamma_0 - \vec{\gamma}\cdot\hat{p})}{D_+(p_0,p)} + \frac{\frac{1}{2}(\gamma_0 + \vec{\gamma}\cdot\hat{p})}{D_-(p_0,p)}. \qquad (2)$$

The arguments below will show that S^R contains four poles. In the fourth quadrant there is a pole at a complex energy $\mathcal{E}_p$ for particle excitations and another at a different complex energy $\mathcal{E}_h$ for hole excitations. The two poles for the corresponding antiparticle and antihole are in the third quadrant.

4. Existence of separate dispersion relations

The first step in the argument is the observation that at $p = 0$ the propagator cannot depend on the direction of the unit vector $\hat{p}$. This is because the dependence on $\hat{p}$ of the fermion self-energy comes entirely from the the vector $\vec{p}$ as shown in Appendix A. Therefore $D_+(p_0, 0) = D_-(p_0, 0)$ and the propagator has the form

$$S^R(p_0, 0) = \frac{\gamma_0}{2D_+(p_0,0)} = \frac{\gamma_0}{2D_-(p_0,0)}. \qquad (3)$$

Now comes the one dynamical input, namely that when $p = 0$ there is a pole in the energy variable at the thermal mass of the quark so that

$$S^R(p_0, 0) = \frac{\gamma_0}{2(p_0 - \mathcal{M})\,a(p_0)}, \qquad (4)$$

where $a(p_0)$ is analytic at $p_0 = \mathcal{M}$. The mass $\mathcal{M}$ is complex with a negative imaginary part since it is a pole of the retarded propagator. Denote this mass by

$$\mathcal{M} = m - i\gamma/2. \qquad (5)$$

From dimensional analysis both m and γ must be proportional to temperature. The real part, m, has been extracted from lattice calculations [41].

The next step is to examine $p \neq 0$. The denominators $D_\pm(p_0, p)$ must have the structure

$$D_+(p_0,p) = (p_0 - \mathcal{M})a(p_0) - b(p_0,p)$$
$$D_-(p_0,p) = (p_0 - \mathcal{M})a(p_0) - c(p_0,p),$$

where b and c both vanish at $p = 0$. Since b and c are continuous in p and vanish at zero momentum, they can be made arbitrarily small by choosing p sufficiently small. Therefore in the neighborhood of $p_0 \approx \mathcal{M}$ and $p \approx 0$ one can approximate

$$D_+(p_0,p) \approx (p_0 - \mathcal{M})a(\mathcal{M}) - b(\mathcal{M},p)$$
$$D_-(p_0,p) \approx (p_0 - \mathcal{M})a\mathcal{M}) - c(\mathcal{M},p).$$

D_+ vanishes at a complex energy $p_0 = \mathcal{E}_p$ given by

$$\mathcal{E}_p = \mathcal{M} + \frac{b(\mathcal{M},p)}{a(\mathcal{M})} + \ldots (p \text{ small}). \qquad (6a)$$

The subscript on $\mathcal{E}_p$ indicates that it is the complex energy of the particle excitation. It is, of course, a function of the momentum p. D_- vanishes at a different energy $p_0 = \mathcal{E}_h$ given by

$$\mathcal{E}_h = \mathcal{M} + \frac{c(\mathcal{M},p)}{a(\mathcal{M})} + \ldots (p \text{ small}), \qquad (6b)$$

which is the complex energy of the collective hole excitation. Note that it is not necessary to assume that b and c are differentiable at $p = 0$. Equations (6a) and (6b) must have different momentum dependence because $b(\mathcal{M}, p) \neq c(\mathcal{M}, p)$ or else the full propagator in Eq. (2) would be completely independent of $\vec{\gamma}\cdot\hat{p}$.

For general momentum p the particle and hole energies are solutions to

$$0 = (\mathcal{E}_p - \mathcal{M})a(\mathcal{E}_p) - b(\mathcal{E}_p, p) \qquad (7a)$$
$$0 = (\mathcal{E}_h - \mathcal{M})a(\mathcal{E}_h) - c(\mathcal{E}_h, p). \qquad (7b)$$

Since these energies are poles of the retarded propagator, they have negative imaginary parts:

$$\mathcal{E}_p = E_p(p) - i\gamma_p(p)/2 \tag{8a}$$
$$\mathcal{E}_h = E_h(p) - i\gamma_h(p)/2. \tag{8b}$$

At zero momentum $E_p(0) = E_h(0)$ and $\gamma_p(0) = \gamma_h(0)$ as indicated in Eq. (5). Although the arguments for this depend only on rotational invariance, perturbative calculations do show these properties. Equality of the zero-momentum energies is displayed in the one-loop calculations [1,3–6,39]; equality of the zero-momentum damping rates was found by Braaten and Pisarski [7] using HTL resummation.

5. Reflection symmetry

The momentum variable p originally has a positive, real value $|\vec{p}|$. However, one can extend the functions $D_\pm(p_0, p)$ to more general values of p. In particular, it is useful to allow p to be real and negative. Appendix B shows that

$$D_-(p_0, p) = D_+(p_0, -p), \tag{9}$$

or equivalently $b(p_0, -p) = c(p_0, p)$. Under the change p to $-p$ in Eq. (7a), let the solution be $\Omega \equiv \mathcal{E}_p(-p)$. Then Eq. (7a) becomes

$$0 = (\Omega - \mathcal{M})a(\Omega) - c(\Omega, p).$$

Comparison with Eq. (7b) shows that $\Omega = \mathcal{E}_h(p)$. This gives the reflection relation

$$\mathcal{E}_p(-p) = \mathcal{E}_h(p). \tag{10}$$

The real and imaginary parts of this are

$$E_p(-p) = E_h(p) \tag{11a}$$
$$\gamma_p(-p) = \gamma_h(p). \tag{11b}$$

At $p = 0$ this coincides with the previous results. Differentiating with respect to p and then setting $p = 0$ gives a new result:

$$\frac{\partial E_p}{\partial p}\Big|_{p=0} = -\frac{\partial E_h}{\partial p}\Big|_{p=0} \tag{12a}$$
$$\frac{\partial \gamma_p}{\partial p}\Big|_{p=0} = -\frac{\partial \gamma_h}{\partial p}\Big|_{p=0}. \tag{12b}$$

(There is a caveat that goes with these derivative relations: It is possible that the derivatives do not exist. For example, one cannot rule out the existence of a branch point at $p = 0$, though there is no reason to expect it.) Peshier and Thoma have obtained Eq. (12a) nonperturbatively using a Ward identity argument [40]. In perturbative calculations the opposite slopes of the real energies is displayed in the one-loop calculations [1,3–6,39] and the opposite slopes of the damping rates is a property of the calculations of Pisarski [8] and of Boyanovsky et al [33] if one extends their answers to $p = 0$. Strictly speaking, the hard thermal loop calculations of the damping rates do not allow zero momentum but are limited to $p > g^2T$. Repeated differentiation of Eq. (11) gives the general relations

$$\frac{\partial^n E_p}{\partial p^n}\Big|_{p=0} = (-1)^n \frac{\partial^n E_h}{\partial p^n}\Big|_{p=0} \tag{13a}$$
$$\frac{\partial^n \gamma_p}{\partial p^n}\Big|_{p=0} = (-1)^n \frac{\partial^n \gamma_h}{\partial p^n}\Big|_{p=0}, \tag{13b}$$

provided the derivatives exist.

The vanishing of D_+ at $p_0 = \mathcal{E}_p$ and of D_- at $p_0 = \mathcal{E}_h$ along with the reflection symmetry (9) means that the retarded propagator has the form

$$S^R(p_0, \vec{p}) = \frac{1}{2}(\gamma_0 - \vec{\gamma}\cdot\hat{p})\Big(\frac{Z_p(p)}{p_0 - \mathcal{E}_p} - d(p_0, p)\Big) + \frac{1}{2}(\gamma_0 + \vec{\gamma}\cdot\hat{p})\Big(\frac{Z_h(p)}{p_0 - \mathcal{E}_h} - d(p_0, -p)\Big). \tag{14}$$

Here d is an unknown function except for the requirement that it has no singularities in the upper half of the complex p_0 plane. The complex residues satisfy the reflection property

$$Z_p(-p) = Z_h(p), \tag{15}$$

which implies

$$Z_p(0) = Z_h(0) \tag{16a}$$
$$\frac{\partial^n Z_p}{\partial p^n}\Big|_{p=0} = (-1)^n \frac{\partial^n Z_h}{\partial p^n}\Big|_{p=0}. \tag{16b}$$

Equality of the residues and of the first derivatives has been found in one-loop perturbation theory [1,3–6].

6. Charge conjugation invariance

The propagator in Eq. (14) is invariant under chirality and parity by construction. Appendix C shows that invariance under time reversal requires Eq. (C12), which is automatically satisfied by the above. However invariance under charge conjugation, Eq. (C13), is not automatic but requires the denominators to satisfy

$$D_-(p_0, p) = -[D_+(-p_0^*, p)]^*. \tag{17}$$

Imposing this on Eq. (14) gives

$$\begin{aligned} S^R(p_0, \vec{p}) &= \frac{1}{2}(\gamma_0 - \vec{\gamma}\cdot\hat{p})\Big(\frac{Z_p(p)}{p_0 - \mathcal{E}_p} + \frac{Z_h^*(p)}{p_0 + \mathcal{E}_h^*} - f(p_0, p)\Big) \\ &+ \frac{1}{2}(\gamma_0 + \vec{\gamma}\cdot\hat{p})\Big(\frac{Z_h(p)}{p_0 - \mathcal{E}_h} + \frac{Z_p^*(p)}{p_0 + \mathcal{E}_p^*} + f^*(-p_0^*, p)\Big). \end{aligned} \tag{18}$$

The new p_0 poles at $-\mathcal{E}_h^*$ and at $-\mathcal{E}_p^*$ are due to the antiparticles of the hole excitation and particle excitation,

respectively. Because $\mathcal{E}_p$ and $\mathcal{E}_h$ are in the fourth quadrant, then $-\mathcal{E}_p^*$ and $-\mathcal{E}_h^*$ are in the third quadrant. The function $f(p_0,p)$ can have no singularities in the upper-half of the complex plane. It must satisfy the reflection property

$$f(p_0,p) = -f^*(-p_0^*,-p), \tag{19}$$

which is observed in the one-loop calculations [1,3–6].

The matrix structure of the propagator comes from the usual Dirac spinors:

$$\frac{1}{2}\big(\gamma_0 - \vec{\gamma}\cdot\hat{p}\big)_{\alpha\beta} = \sum_s u_\alpha(\vec{p},s)\overline{u}_\beta(\vec{p},s) \tag{20a}$$

$$\frac{1}{2}\big(\gamma_0 + \vec{\gamma}\cdot\hat{p}\big)_{\alpha\beta} = \sum_s v_\alpha(-\vec{p},s)\overline{v}_\beta(-\vec{p},s). \tag{20b}$$

Both sets of spinors are eigenstates of $\chi \equiv \gamma_5\vec{\Sigma}\cdot\hat{p} = \gamma_0\vec{\gamma}\cdot\hat{p}$. Since those in Eq. (20a) have $\chi = +1$, the chirality and helicity have the same sign for particles and for anti-holes. The spinors in Eq. (20b) have $\chi = -1$, indicating that the chirality and helicity have the opposite sign for antiparticles and for holes.

7. Additional properties

The advanced propagator is obtained by the relation in Eq. (C5) of Appendix C:

$$\begin{aligned} S^A(p_0,\vec{p}) &\\ = \frac{1}{2}(\gamma_0-\vec{\gamma}\cdot\hat{p})&\Big(\frac{Z_p^*(p)}{p_0-\mathcal{E}_p^*} + \frac{Z_h(p)}{p_0+\mathcal{E}_h} - f^*(p_0^*,p)\Big)\\ +\frac{1}{2}(\gamma_0+\vec{\gamma}\cdot\hat{p})&\Big(\frac{Z_h^*(p)}{p_0-\mathcal{E}_h^*} + \frac{Z_p(p)}{p_0+\mathcal{E}_p} + f(-p_0,p)\Big). \end{aligned} \tag{21}$$

It has four poles in the upper half-plane as well as branch cuts in the upper half-plane from f.

As discussed in Appendix C, the spectral function may be expressed as the difference between the retarded and advanced propagators. It is convenient to define that part of the spectral function proportional to γ_0 as

$$\rho_0(p_0,p) = \frac{1}{4}\mathrm{Tr}[\,\rho(p_0,p)\gamma_0].$$

As shown in Appendix C, ρ_0 must be positive:

$$\begin{aligned}\rho_0(p_0,p) = -\mathrm{Im}\Big[\frac{Z_p(p)2\mathcal{E}_p}{p_0^2-\mathcal{E}_p^2} + \frac{Z_h(p)2\mathcal{E}_h}{p_0^2-\mathcal{E}_h^2}\Big]&\\ +\mathrm{Im}\Big[f(p_0,p)+f(-p_0,p)\Big] > 0,&\end{aligned} \tag{22}$$

The canonical anticommutation relations of the fermion field operators impose require the integral of ρ_0 to be unity as shown in Eq. (C9). This gives

$$\mathrm{Re}\big(Z_p(p)\big) + \mathrm{Re}\big(Z_h(p)\big) + \int_{-\infty}^{\infty}\frac{dp_0}{\pi}\,\mathrm{Im}f(p_0,p) = 1. \tag{23}$$

It is perhaps worth noting that the explicit one-loop calculations satisfy additional sum rules for the first and second moments of the energy, but these apply only at one-loop [5].

The time-ordered propagator can be expressed directly in terms of the retarded and advanced propagators as

$$S^{11}(p_0,\vec{p}) = S^R(p_0,\vec{p})\,e^{\beta p_0}n(p_0) + S^A(p_0,\vec{p})\,n(p_0). \tag{24}$$

It is rather trivial to prove this for p_0 real. Appendix D proves that it holds throughout the complex p_0 plane. Therefore in addition to the kinematic poles coming from the Fermi-Dirac function, the time-ordered propagator has eight dynamical poles: four in the lower half-plane from S^R and four in the upper half-plane from S^A. Of course the dynamical poles are all reflections of the basic particle and hole energies $\mathcal{E}_p$ and $\mathcal{E}_h$.

8. Conclusion

The above results depend only upon invariance under chirality, parity, charge conjugation, and time reversal and not at all upon perturbation theory. The complex energies $\mathcal{E}_p(p)$ and $\mathcal{E}_h(p)$ of the particle and hole excitations are gauge-fixing invariant as proven generally by Kobes, Kunsattter, and Rebhan [42]. However the residue functions Z_p and Z_h are expected to change with gauge. Appendix D shows that the renormalized electric charge for the particle and hole excitations, as measured by the coupling to photons at zero momentum, has the correct value independent of the functions Z_p and Z_h.

The general shape of the two dispersion curves requires difficult calculation. The only simple property is that as $p \to \infty$ the effects of temperature diminish so that $\mathcal{E}_p(p) \to p$ and $Z_h(p) \to 0$. The hard thermal loop calculations enjoy several properties that have not been proven to hold generally. Namely, in the HTL approximation the hole energy is asymptotic to p as $p \to \infty$, the phase velocities of both excitations are larger than unity, and the group velocities of both excitations are smaller than unity.

At temperatures close to the critical temperature the running coupling is large and one-loop calculations do not apply. Since the doubled dispersion relation is such a characteristic feature of the high-temperature, chirally symmetric phase of QCD, it is frustrating that experimental signatures are so difficult to find. One possibility are the Van Hove singularities in dilepton production found by Braaten, Pisarski, and Yuan [12]. This has been recently studied by Schäfer and Thoma [39] and Peshier and Thoma [40]. The Van Hove singularities are determined by the minimum value of the hole dispersion relation $E_{\min}$ and by the maximum energy difference between the two dispersion relations ΔE. In the dilepton rest frame the dilepton rate has a square root divergence at $k_0 = 2E_{\min}$ and at $k_0 = \Delta E$. Unfortunately in the HTL approximation the values $E_{\min} = 0.38gT$ and

$\Delta E = 0.19gT$ are so small that the effect is swamped by the large dilepton continuum. However since the continuum falls rapidly with energy, it is possible that the Van Hove singularities might exceed the continuum if the true values of $E_{\rm min}$ and $E_{\rm max}$ are large enough. Of course, the effect will be weakened by the quark damping rates.

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APPENDIX A: $p = 0$ independent of $\hat{p}$

A crucial step in the proof that particles and holes have separate dispersion relations, which is used in Eq. (3) and subsequently, is the fact that at zero three-momentum the fermion propagator does not depend on the momentum direction $\hat{p}$. (This property is specific to fermions. Gauge boson propagators can depend on unit vectors at zero momentum, as occurs even in the free Coulomb-gauge propagator.) The argument is simple. The self-energy at any order of perturbation theory is expressed in terms of momentum integrations over integrands composed of fermion propagators and boson propagators. The routing of the external three momenta $\vec{p}$ through the internal propagators depends on what choice is made for the loop momenta $\vec{k}_j$ that must be integrated. Because of fermion number conservation at each vertex, the external fermion number can be uniquely traced through a sequence of internal fermion propagators that form a continuous path through any self-energy diagram. Therefore one can always choose the loop momenta so that the gauge boson propagators and the fermion propagators in closed loops will contain various $\vec{k}_j$ but never $\vec{p}$. The argument of the linked fermion propagators that connect with the external lines will be the sum of $\vec{p}$ with a linear combination of the $\vec{k}_j$. These internal fermion propagators depend on the vector $\vec{p}$ and not separately on $\hat{p}$ and $|\vec{p}|$. Hence if the external momenta $|\vec{p}|$ is set equal to zero, there can be no dependence on the direction $\hat{p}$ and the propagator must be as shown in Eq. (3).

APPENDIX B: Symmetry under $p \to -p$

It is convenient to parameterize $\vec{p}$ by p, θ, ϕ:

$$\begin{aligned} p_x &= p\sin\theta\cos\phi \\ p_y &= p\sin\theta\sin\phi \\ p_z &= p\cos\theta. \end{aligned}$$

Under the transformation $p \to -p, \theta \to \pi - \theta, \phi \to \phi + \pi$ the vector $\vec{p}$ is unchanged. This amounts to $\hat{p} \to -\hat{p}$ and $p \to -p$. If the propagator is considered as a function of the four variables p_0, p, and $\hat{p}$, it must not change:

$$S^R(p_0, p, \hat{p}) = S^R(p_0, -p, -\hat{p}).$$

In terms of the definitions in Eq. (2) this requires

$$D_-(p_0, p) = D_+(p_0, -p), \tag{B1}$$

as employed in Eq. (9).

APPENDIX C: Retarded propagator

A natural starting point for discussion of any finite-temperature propagator is the finite-temperature spectral function:

$$\rho_{\alpha\beta}(x) = \sum_n e^{-\beta E_n} \frac{\langle n|\{\psi_\alpha(x), \overline{\psi}_\beta(0)\}|n\rangle}{\mathrm{Tr}[e^{-\beta H}]}, \tag{C1}$$

where the sum is over a complete set of energy eigenstates $|n\rangle$. In the following, Dirac indices will be suppressed and ρ will be treated as a matrix. The matrix satisfies:

$$[\rho(x)]^\dagger = \gamma_0 \rho(-x) \gamma_0. \tag{C2}$$

1. *Retarded vs advanced*

The basic retarded and advanced propagators are defined in coordinate space as

$$\begin{aligned} S^R(x) &= -i\theta(x_0)\rho(x) \\ S^A(x) &= i\theta(-x_0)\rho(x). \end{aligned}$$

In momentum space they have the simple dispersion relations

$$S^R(p_0, \vec{p}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \frac{\rho(\omega, \vec{p})}{p_0 - \omega + i\eta} \tag{C3}$$

$$S^A(p_0, \vec{p}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \frac{\rho(\omega, \vec{p})}{p_0 - \omega - i\eta}. \tag{C4}$$

The retarded propagator is an analytic function of complex p_0 in the half-plane $\mathrm{Im}p_0 > 0$. Its only singularities are in the lower half-plane. Similarly the advanced propagator is an analytic function of complex p_0 in the half-plane $\mathrm{Im}p_0 < 0$. All its singularities are in the upper half-plane. The two propagators are connected by the relation

$$S^A(p_0^*, \vec{p}) = \gamma_0 [S^R(p_0, \vec{p})]^\dagger \gamma_0, \tag{C5}$$

which follows directly from the adjoint property of the spectral function in Eq. (C2).

2. Spectral Function

Using Eqns. (C3) and (C4) the spectral function in momentum space can be expressed as

$$\rho(\omega, p) = i\big[S^R(\omega, \vec{p}) - S^A(\omega, \vec{p})\big]. \tag{C6}$$

Although the spectral function is defined for real ω, this relation allows ω to be continued into the complex plane. From the definition of the spectral function the matrix $[\rho(\omega, \vec{p})\gamma_0]_{\alpha\beta}$ is Hermitian and has a positive-definite trace:

$$\rho_0(\omega, \vec{p}) \equiv \frac{1}{4}\mathrm{Tr}[\rho(\omega, \vec{p})\gamma_0] > 0. \tag{C7}$$

This leads to the condition in Eq. (22).

From the canonical anticommutation relations at equal times, the spectral function must satisfy the relation $\gamma_0\delta^3(\vec{x}) = \rho(0, \vec{x})$. The Fourier transform is

$$\gamma_0 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \rho(\omega, \vec{p}).$$

From Eq. (C6) this can be expressed as

$$\gamma_0 = i\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \big(S^R(\omega, \vec{p}) - S^A(\omega, \vec{p})\big). \tag{C8}$$

When Eq. (18) and (21) are used the terms proportional to $\vec{\gamma}\cdot\hat{p}$ automatically integrate to zero and those proportional to γ_0 give the sum rule

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \rho_0(\omega, p) = 1. \tag{C9}$$

3. Discrete symmetries

In the high temperature phase, QCD is expected to be invariant under chirality as well as under parity, charge conjugation, and time reversal. These symmetries constrain the spectral function and consequently the propagators.

(i) Chirality invariance requires that the spectral function satisfy $\rho(x) = -\gamma_5\rho(x)\gamma_5$. For the retarded propagator this implies that

$$Q_5: \qquad S^R(p_0, \vec{p}) = -\gamma_5 S^R(p_0, \vec{p})\gamma_5. \tag{C10}$$

(ii) Parity requires $\rho(t, \vec{x}) = \gamma_0\rho(t, -\vec{x})\gamma_0$ and therefore

$$P: \qquad S^R(p_o, \vec{p}) = \gamma_0 S^R(p_0, -\vec{p})\gamma_0. \tag{C11}$$

(iii) Time reversal invariance requires $[\rho(t, \vec{x})]^* = T\rho(-t, \vec{x})T$ where $T = i\gamma^1\gamma^3$ and relates the retarded propagator to the advanced:

$$[S^R(p_0, \vec{p})]^* = T S^A(p_0^*, -\vec{p})T.$$

Using Eq. (C5) this can be expressed in terms of S^R:

$$T: \qquad S^R(p_0, \vec{p}) = T\gamma_0[S^R(p_0, -\vec{p})]^T\gamma_0 T. \tag{C12}$$

This relation is automatically satisfied by Eq. (14).

(iv) Charge conjugation requires $[\rho(x)]^T = C\rho(-x)C$ where $C = i\gamma^2\gamma^0$. This relates the retarded propagator to the advanced:

$$[S^R(p_0, \vec{p})]^T = -C\, S^A(-p_0, -\vec{p})\, C.$$

However, using Eq. (C5) this can be rewritten as a constraint on the retarded propagator directly:

$$C: \qquad S^R(p_0, \vec{p}) = -C\gamma_0[S^R(-p_0^*, -\vec{p})]^*\gamma_0 C. \tag{C13}$$

It is this relation that leads from Eq. (14) to Eq. (18).

(v) It is sometimes convenient to deal with the combination $\theta = TCP$, which is always an invariance. Thus $[\rho(x)]^\dagger = -\gamma_0\gamma_5\rho(x)\gamma_5\gamma_0$ always holds and therefore

$$TCP: \qquad [S^R(p_0, \vec{p})]^\dagger = \gamma_5\gamma_0 S^R(-p_0^*, -\vec{p})\gamma_0\gamma_5 \tag{C14}$$

must always hold.

APPENDIX D: Time-ordered propagator

This appendix proves Eq. (24), which expresses the time-ordered propagator in momentum space directly in terms of the retarded and advanced propagators. The relation holds for all p_0. The time-ordered propagator is defined as

$$S^{(11)}_{\alpha\beta}(x) = -i\sum_n e^{-\beta E_n}\frac{\langle n|T(\psi_\alpha(x), \overline{\psi}_\beta(0))|n\rangle}{\mathrm{Tr}[e^{-\beta H}]}.$$

This can be expressed in terms of the spectral function:

$$S^{11}(p_0, \vec{p}) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\Big(\frac{\rho(\omega, \vec{p})\,[1 - n(\omega)]}{p_0 + i\eta - \omega} + \frac{\rho(\omega, \vec{p})\, n(\omega)}{p_0 - i\eta - \omega}\Big),$$

where $n(\omega)$ is the Fermi-Dirac distribution function:

$$n(\omega) = \frac{1}{\exp(\beta\omega) + 1}.$$

It is rather easy to evaluate S^{11} for p_0 real and η infinitesimal and obtain Eq. (24) for real p_0. However, since the focus of the present paper is the pole structure of the propagators in the complex p_0 plane, that easy argument is not adequate. In order for Eq. (24) to hold in the complex p_0 plane it will be important that $n(\omega)$ have not branch points on the real axis.

The dispersion relation for S^{11} shows it to be the sum of two functions. The first is analytic in $\mathrm{Im}p_0 > 0$ and the second is analytic in $\mathrm{Im}p_0 < 0$. Write S^{11} as

$$S^{11}(p_0, \vec{p}) = S^R(p_0, \vec{p}) + F(p_0, \vec{p}) + G(p_0, \vec{p}),$$

where

$$\mathrm{Im}\, p_0 > 0: \qquad F(p_0,\vec{p}) = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\rho(\omega,\vec{p})\, n(\omega)}{p_0-\omega}$$
$$\mathrm{Im}\, p_0 < 0: \qquad G(p_0,\vec{p}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\rho(\omega,\vec{p})\, n(\omega)}{p_0-\omega}.$$

Using Eqns. (C6) for the spectral function, F can be computed by closing the ω contour in the upper half-plane for the term containing S^R and closing it in the lower half-plane for the term containing S^A. Since $n(\omega)$ has poles at $\omega = \pm\Omega_\ell$ where $\Omega_\ell = i\ell\pi T$ the result for F is

$$-S^R(p_0,\vec{p})n(p_0) - T\sum_{\ell=1}^{\infty}\Big(\frac{S^R(\Omega_\ell,\vec{p})}{p_0-\Omega_\ell} + \frac{S^A(-\Omega_\ell,\vec{p})}{p_0+\Omega_\ell}\Big).$$

This is analytic for $\mathrm{Im}\, p_0 > 0$ but has poles and cuts for $\mathrm{Im}\, p_0 < 0$. In the same fashion G can be computed by closing the ω contour in the upper half-plane for the term containing S^R and closing in the lower half-plane for the term containing S^A with the result

$$T\sum_{\ell=1}^{\infty}\Big(\frac{S^R(\Omega_\ell,\vec{p})}{p_0-\Omega_\ell} + \frac{S^A(-\Omega_\ell,\vec{p})}{p_0+\Omega_\ell}\Big) + S^A(p_0,\vec{p})n(p_0).$$

This is analytic for $\mathrm{Im}\, p_0 < 0$ but has poles and cuts in $\mathrm{Im}\, p_0 > 0$. When the results for F and G are added the sums over ℓ coming from the poles of the Fermi-Dirac function $n(\omega)$ cancel and give

$$F(p_0,\vec{p}) + G(p_0,\vec{p}) = \big(-S^R(p_0,\vec{p}) + S^A(p_0,\vec{p})\big)\, n(p_0).$$

The final result for the time-ordered propagator is

$$S^{11}(p_0,\vec{p}) = S^R(p_0,\vec{p})\,[1-n(p_0)] + S^A(p_0,\vec{p})\, n(p_0).$$

Since $1-n(p_0) = e^{\beta p_0} n(p_0)$ the result can be expressed as Eq. (24).

APPENDIX E: Electric charge

This appendix will show that the four poles in the retarded propagator (18) all have the correct value of the renormalized electric charge. The operator ψ for the u quark destroys charge $e_\psi = 2e/3$; the operator ψ for the d quark destroys charge $e_\psi = -e/3$. The amplitude for a photon with zero four-momentum to couple to a fermion is

$$S_R(p_0,\vec{p})\Gamma_\mu(p_0,\vec{p})S_R(p_0,\vec{p}).$$

At finite temperature the Ward identity

$$\Gamma_\mu = \frac{\partial}{\partial p^\mu}\big[S^R(p_0,\vec{p})\big]^{-1}.$$

determines the photon vertex function [44–46]. The inverse of the propagator in Eq. (2) can be written

$$[S(p_0,\vec{p})]^{-1} = \gamma_0 A(p_0,p) - \vec{\gamma}\cdot\hat{p}B(p_0,p),$$

where $D_\pm = A \mp B$. The Γ_0 vertex is thus

$$\Gamma_0(p_0,\vec{p}) = \gamma_0 A'(p_0,p) - \vec{\gamma}\cdot\hat{p}B'(p_0,p), \qquad \text{(E1)}$$

where prime denotes the p_0 derivative.

1. Pole at $\mathcal{E}_p$ or $-\mathcal{E}_h^*$

When $p_0 \to \mathcal{E}_p$ the retarded propagator, Eq. (18), is dominated by the pole

$$S^R(p_0,\vec{p}) \to Z_p(p)\frac{\sum_{\rm spin} u(\vec{p})\overline{u}(\vec{p})}{p_0-\mathcal{E}_p}.$$

If the quark field ψ carries bare electric charge $e_{0\psi}$, then the renormalized electric charge which couples the photon to the particle excitation is

$$Q = e_{0\psi}\sqrt{Z_3}Z_p(p)\cdot\overline{u}(\vec{p})\Gamma_0(p_0,\vec{p})u(\vec{p}),$$

where $\sqrt{Z_3}$ comes from the photon propagator and $\sqrt{Z_p}$ comes from each of the two fermion fields at the vertex. Substituting Eq. (E1) gives

$$Q = e_{0\psi}\sqrt{Z_3}Z_p(p)\big[A'(p_0,p) - B'(p_0,p)\big]_{p_0=\mathcal{E}_p}.$$

As $p_0 \to \mathcal{E}_p$ the denominator $D_+ = A - B$ vanishes linearly: $A - B \to (p_0-\mathcal{E}_p)/Z_p(p)$, so that

$$Q = e_{0\psi}\sqrt{Z_3} = e_\psi. \qquad \text{(E2)}$$

It is easy to check that the pole in Eq. (18) at $p_0 = -\mathcal{E}_h^*$ also couples with the same charge (E2).

2. Pole at $\mathcal{E}_h$ or $-\mathcal{E}_p^*$

At the hole excitation, $p_0 \to \mathcal{E}_h$, the retarded propagator, Eq. (18), behaves as

$$S^R(p_0,\vec{p}) \to Z_h(p)\frac{\sum_{\rm spin} v(-\vec{p})\overline{v}(\vec{-p})}{p_0-\mathcal{E}_h}.$$

The renormalized electric charge of the hole excitation is

$$Q = e_{0\psi}\sqrt{Z_3}Z_h(p)\cdot\overline{v}(-\vec{p})\Gamma_0(p_0,\vec{p})v(-\vec{p}).$$

Substituting from Eq. (E1) gives

$$Q = e_{0\psi}\sqrt{Z_3}Z_h(p)\big[A'(p_0,p) + B'(p_0,p)\big]_{p_0=\mathcal{E}_h}.$$

At the hole excitation, $D_- = A + B \to (p_0-\mathcal{E}_h)/Z_h(p)$ as $p_0 \to \mathcal{E}_h$ so that

$$Q = e_{0\psi}\sqrt{Z_3} = e_\psi. \qquad \text{(E3)}$$

The same argument applies to the pole in Eq. (18) at $p_0 = -\mathcal{E}_p^*$.

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